Radial solutions to elliptic equations

Alfonso Castro

Harvey Mudd College

- - -

Kennesaw State University, November 2020

- - -

The author thanks Professor Jorge Cossio for his careful reading of the original presentation, pointing out multiple errors, and thoughtful suggestions.

Hessian of radial functions

The Hessian of a function u at x is given by the matrix

$$D^2(u)(x) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right). \tag{1}$$

If u is a radial function and T is a rotation ($u(x) = u(Tx)$) we have $D^2u(x) = T^t D^2 u(Tx) T$. Hence, if $\|x| = \|y\|$, then $D^2u(x)$ and $D^2u(y)$ have the same invariants (eigenvalues, trace, ..., determinant). In particular, if $r = \|x\|$ we see that the invariants of $D^2u(x)$ are the invariants of

$$D^2u(r,0,\dots,0) = \begin{pmatrix} u_{rr} & 0 & 0 & \cdots & 0 \\ 0 & \frac{u_r}{r} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \frac{u_r}{r} \end{pmatrix} (r,0,\dots,0). \tag{2}$$

Invariants of the Hessian

Computing the trace of $D^2u(r,0,\dots,0)$ we have

$$\Delta u(x) = u_{rr}(r,0,\dots,0) + \frac{N-1}{r}u_r(r,0,\dots,0). \tag{3}$$

The k-th invariant of $D^2u(r,0,\dots,0)$ is the given then by

$$\begin{aligned} S_k u(x) &= \binom{N-1}{k-1}\left(\frac{u_r(r)}{r}\right)^{k-1} u_{rr}(r) + \binom{N-1}{k}\left(\frac{u_r(r)}{r}\right)^{k} \\ S_N u(x) &= \left(\frac{u_r(r)}{r}\right)^{N-1} u_{rr}(r), \end{aligned} \tag{4}$$

The operator S_N is known as the Monge-Ampere operator.

The p-Laplacian operator in $\mathbb{R}^N$

For $p > 0$ and radial functions u, the p-Laplacian operator is defined by

$$\Delta_p u(r) = \left(|u_r(r)|^{p-2} u_r(r)\right)_r + \frac{N-1}{r} |u_r(r)|^{p-2} u_r(r). \tag{5}$$

For radial functions the p-Laplacian operator and the k-Hessian operators are particular cases of operators of the form

$$Q(u)(r) = r^{-\gamma} (r^{\alpha} |u_r(r)|^{\beta} u_r(r))_r. \tag{6}$$

Motivating example

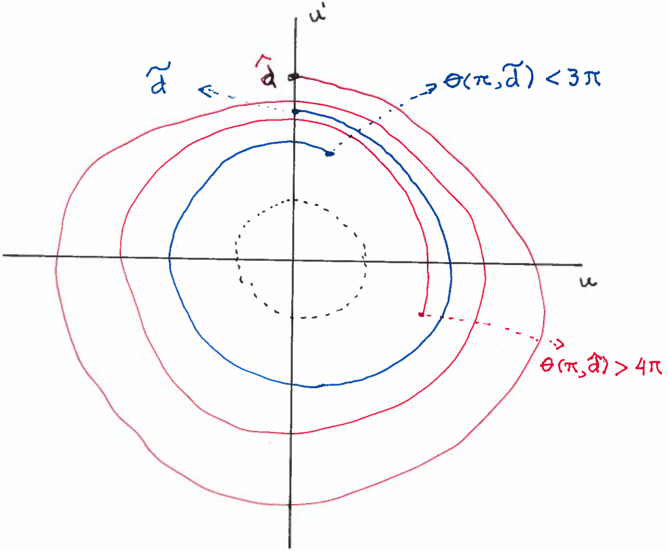
$$u''(t) + g(u(t)) = q(t), \;\; t \in (0, \pi) \quad u(0) = 0, \;\; u(\pi) = 0. \tag{7}$$

$$\sigma(-'') = \{1, 4, \ldots, k^2, \ldots \to +\infty\}, \tag{8}$$
All the eigenvalues are simple.

THEOREM 1.1 *If*

$$\lim_{|u|\to+\infty} \frac{g(u)}{u} = +\infty \quad (\textit{superlinear}), \tag{9}$$

g is differentiable, and q is bounded and continuous then (7) has infinitely many solutions. Moreover, there exists K such that if $k \geq K$ is a positive integer then (7) has two solutions with k nodes (zeroes).


u'
u
$\tilde{d}$
$\hat{d}$
$\theta(\pi,\tilde{d}) < 3\pi$
$\theta(\pi,\hat{d}) > 4\pi$

Proof of Theorem 1.1. Let $u(\cdot, d)$ satisfy $u''(t) + g(u(t)) = Q(t)$ $t \in (0, \pi)$, $u(0) = 0$, $u'(0) = d$. Let $G(u) = \int_0^u g(s)ds$ and

$$E(t, d) = \frac{(u'(t, d))^2}{2} + G(u(t, d)). \tag{10}$$

Using that G is bounded below,

$$\begin{aligned} E'(t, d) &= u'(t)u''(t) + g(u(t))u'(t) = Q(t)u'(t) \\ &\geq -\frac{|u'(t)|^2}{8} - 2\|Q\|_\infty^2 \\ &\geq -\frac{1}{4}E(t, d) - K. \end{aligned} \tag{11}$$

Hence

$$E(t, d) \geq e^{-t/4}\left(\frac{(u'(0))^2}{2} - 4Ke^{1/4}\right) \to +\infty, \tag{12}$$

as $|d| = |u'(0)| \to +\infty$ uniformly for $t \in [0, 1]$.

Therefore, there exists D and a continuous function $\theta(t,d)$ such that

$$
\begin{aligned}
& u(t,d) = \rho(t,d)\sin(\theta(t,d)),\ \ u'(t,d) = \rho(t,d)\cos(\theta(t,d)) \\
& \lim_{|d|\to+\infty} \theta(\pi,d) = +\infty.
\end{aligned} \tag{13}
$$

for $d > D$. Here $\rho^2(t,d) = u^2(t,d) + (u'(t,d))^2$. Similarly for $d < -D$.

Problem

Let Ω be bounded region in $\mathbb{R}^N$.

$$\Delta u(x) + g(u(x)) = f(x),\ x \in \Omega \quad u(x) = 0,\ x \in \partial\Omega. \tag{14}$$

$$\sigma(-\Delta) = \{\lambda_1, \dots, \lambda_k, \dots \to +\infty\}, \tag{15}$$

The eigenvalue λ_1 is a simple, others have finite multiplicity but need not be simple.

PROBLEM. *Suppose g is superlinear, differentiable, and f is bounded and continuous. Does (14) have infinitely many solutions? Are there solutions with large number of nodal regions?*

From (13), there exists a sequence $d_n \to +\infty$ such that $\theta(\pi, d_n) = n\pi$.
That is $u(\cdot, d_n)$ is a solution to (7) with $n-1$ zeroes in $(0, \pi)$.

ANSWER TO PROBLEM $=$ NO, S. Pohozaev (1965).

THEOREM 1.2. (Pohozaev identity) *If u be a solution to* (14) *with* $f = 0$*, then*

$$\int_{\Omega} \left(NG(u) - \frac{N-2}{2} ug(u) \right) dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (\eta(x) \cdot x) dS. \tag{16}$$

Proof. Multiply by u and integrate by parts eliminating integrals of second order derivatives. Multiply by $x \cdot \nabla u$ and integrate by parts eliminating terms that contain second order derivatives. Combining the resulting relations (16) appears by eliminating the terms that contain derivatives of u integrated on Ω .

THEOREM 1.3. *If $g(u) = |u|^{p-1}u$, $p \geq (N+2)/(N-2)$, $f = 0$, and Ω is starlike then $u = 0$ is the only solution to* (14).

Proof. Now $NG(u) - \frac{N-2}{2}ug(u) = \gamma_p|u|^{p+1}$ with $\gamma_p = \frac{2N-(p+1)(N-2)}{2N}$. Hence $\gamma_p \leq 0$. This, (16), and $\eta(x) \cdot x \geq 0$ prove that if u is a solution to (14) then the left hand side in (16) is not positive while the right hand side is not negative. Hence $u = 0$.

Note. a) Imitating the proof of Theorem 1.1 one sees that if $0 < a < b < \infty$,

$$\Omega = \{x \in \mathbb{R}^N; a < \|x\| < b\} := A. \tag{17}$$

then (14) has infinitely many radial solutions. See [B-C, 1988, P-1989].
b) Theorem 1.3 applies to the case

$$\Omega = \{x \in \mathbb{R}^N; \|x\| < 1\} := B.$$

RADIAL SOLUTIONS TO (14) IN B

Recall that, in spherical coordinates $(r, \theta_1, \ldots, \theta_{N-1})$,

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} D(\theta_1, \ldots, \theta_{N-1}) u.$$

Thus for radial solutions in B, (14) becomes

$$u_{rr} + \frac{N-1}{r} u_r + g(u(r)) = f(r), u'(0) = 0, u(1) = 0. \tag{18}$$

Trying to imitate the proof of Theorem 1.1 for (18) one needs to consider

$$u_{rr} + \frac{N-1}{r} u_r + g(u(r)) = f(r), u'(0) = 0, u(0) = d, \tag{19}$$

and try to find d such that $u(1, d) = 0$.

This is known as **a shooting argument.** For Theorem 1.1 or the case $\Omega = A$, see (17), one is shooting from a regular point to another regular point of an ordinary differential equation. Now we consider the cases:

- Shooting form a singular point to a regular point (equation (18)).
- Shooting from a regular point to a singular point. Here we obtain singular solutions for (18)
- Shooting from a singular point to a singular point. Here we obtain rotationally invariant solutions to (14) in manifolds of revolution.

Shooting from a singular point to a regular point.

In the study of (18) we distinguish three cases.

- Under Serrin, $\lim_{u\to+\infty}\frac{g(u)}{u^p}\in(0,\infty)$ with $1<p<\frac{N}{N-2}$, $\lim_{u\to-\infty}\frac{g(u)}{u}=+\infty$ (includes subcritical and supercritical behavior at $-\infty$).
- Between Serrin and Sobolev, $\lim_{u\to+\infty}\frac{g(u)}{u^p}\in(0,\infty)$ with $\frac{N}{N-2}<p<\frac{N+2}{N-2}$, and $\lim_{u\to-\infty}\frac{g(u)}{u^q}\in(0,\infty)$ with $1<q<\frac{N+2}{N-2}$
- Sub-super critical case, $1<p<\frac{N+2}{N-2}<q$.

Consider

$$u_{rr} + \frac{N-1}{r} u_r + g(u(r)) = f(r), u'(0) = 0, u(0) = d, \tag{20}$$

From now on $f = 0$, $g(0) = 0$, and g is monotonically increasing.
Additional zeros of g give rise to additional branches of solutions with L_∞ norm bounded between those zeros, see [C-C, 1994].
Energy dissipates rapidly near 0.

$$\begin{aligned} \left(\frac{(u'(t))^2}{2} + G(u(t)) \right)' &= u'(t)u''(t) + g(u(t))u'(t) \\ &= -\frac{N-1}{r}(u'(t))^2 \\ &\leq 0. \end{aligned} \tag{21}$$

Pohozaev identity

Multiplying (20) by $r^{N-1}u$ and integrating on $[0, r]$ we have

$$r^{N-1}u'u - \int_0^r s^{N-1}(u')^2 ds + \int_0^r s^{N-1}u(s)g(u(s))ds = 0 \tag{22}$$

Multiplying (20) by $r^N u'$ and integrating on $[0, r]$ we have

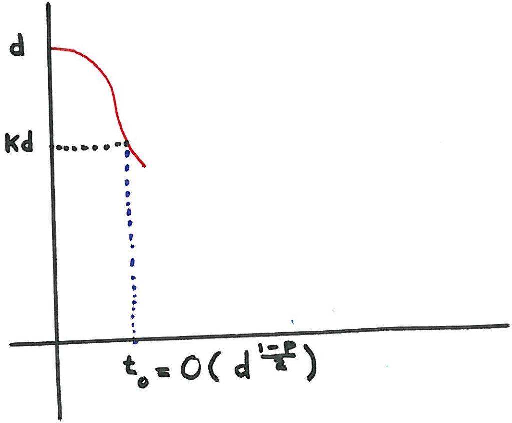
$$\begin{aligned} 0 &= r^N \frac{(u')^2}{2} + (N-1)\int_0^r s^{N-1}(u')^2 ds + \int_0^r s^N (G(u(s)))' ds \\ &= r^N \frac{(u')^2}{2} + r^N G(u(r)) + (N-1)\int_0^r s^{N-1}(u')^2 ds \\ &\quad - \int_0^r N s^{N-1}(G(u(s)))ds \end{aligned} \tag{23}$$

Multiplying (22) by $N-2$ and adding to (23) we have:

$$P(r,d) := r^N\frac{(u')^2}{2} + r^N G(u(r)) + \frac{N-2}{2}r^{N-1}u(r)u'(r) = \int_0^r s^{N-1}\left((NG(u(s)) - \frac{N-2}{2}u(s)g(u(s))\right)ds \tag{24}$$

If $g(u) = |u|^{p-1}u$ then

$$NG(u(s)) - \frac{N-2}{2}u(s)g(u(s)) = \left(\frac{N}{p+1} - \frac{N-2}{2}\right)|u|^{p+1}. \tag{25}$$



d
Kd
$t_0 = O(d^{\frac{1-p}{2}})$

Under Serrin

Since $E' = -\frac{N-1}{r}(u'(r))^2 \geq -\frac{2(N-1)}{r}E(r)$, If $p < \frac{N}{N-2}$ then

$$E(t) \geq \frac{E(t_0)t_0^{2(N-1)}}{t^{2(N-1)}} \geq Kd^{p+1}d^{(1-p)(N-1)} \to +\infty$$

as $d \to +\infty$. The proof follows as in Theorem 1.1

Between Serrin and Sobolev

$$\begin{aligned} P(t_0, d) &= t_0^N E(t_0) + \frac{N-2}{2} t_0^{N-1} u(t_0) u'(t_0) \\ &\geq \left(\frac{N}{p+1} - \frac{N-2}{2}\right) \int_0^{t_0} s^{N-1} u^{p+1}(s) ds \\ &\geq K t_0^N d^{p+1} \geq K d^{(N(1-p)/2)+p+1} \\ &\to +\infty \ \text{ as } d \to +\infty. \end{aligned}$$

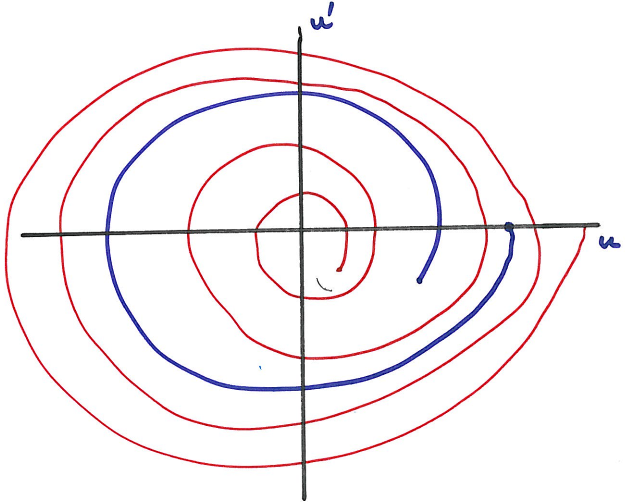
With $p, q \in (1, (N+2)/(N-2))$, $NG(u) - \frac{N-2}{2} u g(u)$ is bounded below. Hence

$$\begin{aligned} E(1) &\geq t_0^N E(t_0) + \frac{N-2}{2}(t_0^{N-1}(u \cdot u')(t_0) - (u \cdot u')(1)) - M \\ &\geq P(t_0, d) - M - \frac{(u'(1))^2}{4} - \frac{(N-2)^2}{2} u^2(1) \end{aligned}$$

Hence $\lim_{d \to +\infty} E(1, d) = +\infty$. Since $E(\cdot, d)$ is a decreasing function we have

$$(26) \qquad \lim_{d \to +\infty} E(t, d) = +\infty \text{ uniformly for } t \in [0, 1].$$

and the existence of infinitely many solutions follows as in Theorem 1.1


u'
u

Nonlinearities with indefinite weight

Recent work with J. Cossio, S. Herrón, and C. Vélez, [C-C-H-V, 2020]. Consider now

$$u_{rr} + \frac{N-1}{r} u_r + W(r)g(u(r)) = f(r), u'(0) = 0, u(1) = 0. \tag{27}$$

where $W : [0,1] \to \mathbb{R}$ is a differentiable function such that there exists $X \in (0,1)$ with $W > 0$ on $[0,X)$, W is negative on $(X,1]$ and $W'(X) < 0$. If g is subcritical then on $[0,X]$ the arguments above yield functions $\rho(r,d)$, $\theta(r,d)$ defined on $([0,X],(D,+\infty))$ such that $\lim_{d\to+\infty} \theta(X,d) = +\infty$.

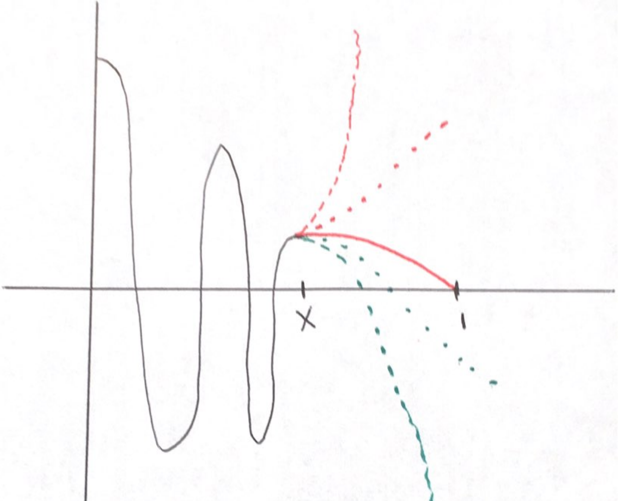
The main difficulty here is that the solutions to

$$u_{rr} + \frac{N-1}{r} u_r + W(r)g(u(r)) = 0, u(X) = a, u'(X) = b \tag{28}$$

may blow up in $(X,1]$.

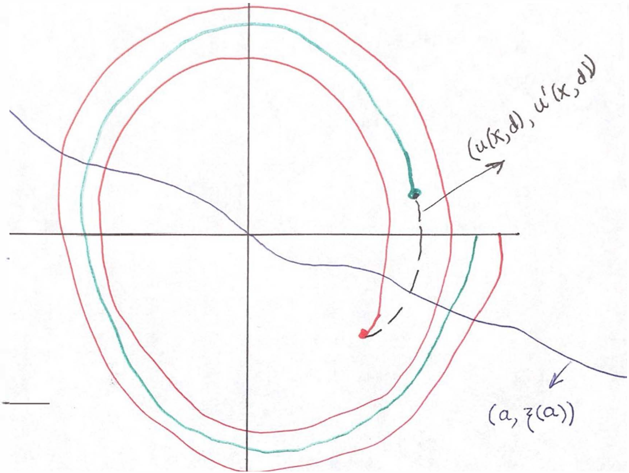
Theorem

If $\lim_{u\to+\infty} \frac{g(u)}{u^p} \in (0,\infty)$, $\lim_{u\to-\infty} \frac{g(u)}{-|u|^q} \in (0,\infty)$, $1 < p, q < (N+2)/(N-2)$ *then* (27) *has infinitely many solutions.*


x

Theorem

There exists a continuous function $\zeta : \mathbb{R} \to \mathbb{R}$ *with* $\zeta(0) = 0$ *such that the solution to* (28) *with* $b = \zeta(a)$ *satisfies* $u(1) = 0$. *Moreover,* u *is positive in* $[X, 1)$ *if* $a > 0$ *and* u *is negative in* $[X, 1)$ *is* $a < 0$.


(u(x,d), u'(x,d))
(a, ξ(a))

The sub-super critical case: $1 < p < (N+2)/(N-2) < q$

$$(29)\qquad \begin{aligned} & t^N E(t) + \frac{N-2}{2} t^{N-1} u(t) u'(t) \geq \\ & t_0^N E(t_0) + \frac{N-2}{2} t_0^{N-1} u(t_0) u'(t_0) - M. \quad \to +\infty \end{aligned}$$

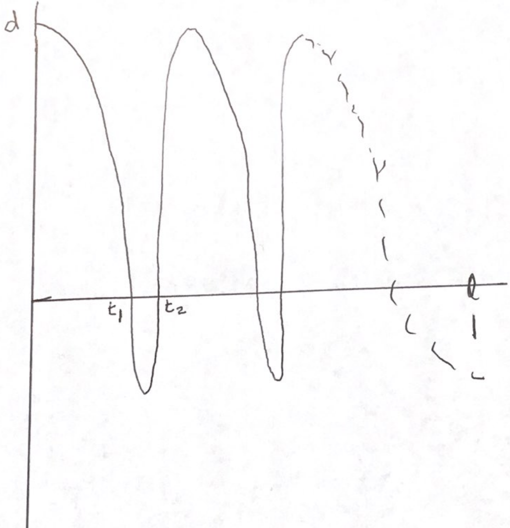
Lemma 1.1 For d large, there exist $t_1 < s_1 < t_2 < s_2$ such that

$$(30)\qquad t_1 > t_0,\ u(t_1) = u'(s_1) = u(t_2) = u'(s_2) = 0,\ s_2 = O(t_0)$$

u decreases on (t_0, s_1), and increases on (s_1, s_2).
Lemma 1.2 For $d > 0$ large,

$$(31)\qquad t_2 - t_1 \leq C_2\, d^{(p+1)\left[\frac{1}{q+1} - \frac{1}{2}\right]} \leq C d^{(1-p)/2}$$

with $C > 0$ independent of d.



d
t_1
t_2

From Lemma 1.2 we have the following.

Lemma 1.3 For $d > 0$ large,

(32) $$P(t,d) \geq P(t_0,d) \text{ for all } t > t_0.$$

Let k be such that $kd^{(1-p)/2} \ll 1$. Repeating this argument k times, we see that there exists $t_1 < t_2 < \cdots < t_k$ such that $u(t_i) = 0$ for $i = 1, 2, \ldots, k$ and $P(t,d) \geq P(t_0,d)$ for all $t \in [t_0, t_k]$. Hence, $\lim_{d\to+\infty} \theta(1,d) = \infty$. Thus:

THEOREM 1.4 If is g is a sub-super critical nonlinearity then (18) has infinitely many solutions.

SINGULAR SOLUTIONS

Shooting from a regular point to a singular and a regular point

Let $N/(N-2) < p < (N+2)/(N-2)$ and $q > 1$.

Theorem 2.1. ([A-C-C-2010]) *If g is subcritical or sub-super critical then* (18) *has a countable number of non-degenerate continua of singular radial solutions.*

The case $p = q$ is found in [D-E-R-2002] for $f = 0$.

SINGULAR SOLUTIONS
Shooting from a regular point to a singular and a regular point

Let $N/(N-2) < p < (N+2)/(N-2)$ and $q > 1$.

Theorem 2.1. ([A-C-C-2010]) *If g is subcritical or sub-super critical then* (18) *has a countable number of non-degenerate continua of singular radial solutions.*

The case $p = q$ is found in [D-E-R-2002] for $f = 0$.

Let

$$(33)\quad \theta = \frac{2}{1-p}, \quad A = (-\theta(\theta + N - 2))^{\frac{1}{p-1}}, \quad \text{and} \quad \tau_2 \in \left(\theta, \frac{2-N}{2}\right).$$

Since $p > N/(N-2)$, there exists $c > 0$ such that

$$(34)\quad \frac{(\theta A + \tau_2 c)^2}{2} + \frac{(A+c)^{p+1}}{p+1} + \frac{(N-2)(A+c)(\theta A + \tau_2 c)}{2} = 0.$$

Let $\tau_1 \in (\theta, \tau_2)$ be such that

$$(35)\qquad (N - 2 + \theta)(A\theta + c\tau_2) + (A+c)^p - c\tau_1(\theta - \tau_2) \neq 0.$$

Let $\tilde{b} = b^{1/(\theta-\tau_2)}$.

Lemma 2.1. Let $I \subset \mathbb{R}$ be a compact interval. For $b \geq 1$ and $a \in I$ there exists a unique function $u(\cdot, a, b) : (0, 1] \to \mathbb{R}$ that satisfies

$$(36) \qquad \begin{aligned} u'' + \frac{N-1}{r} u' + g(u) = 0, \quad 0 < r \leq 1, \\ u(\tilde{b}, a, b) = (A + c)\tilde{b}^{\theta} + a\tilde{b}^{\tau_1}, \text{ and} \\ u'(\tilde{b}, a, b) = (\theta A + \tau_2 c)\tilde{b}^{\theta-1} + \tau_1 a\tilde{b}^{\tau_1 - 1}. \end{aligned}$$

Moreover, $u(x) = u(\|x\|, a, b) \in H^{1,1}(B)$ satisface $\Delta u + g(u) = 0$ as a distribution, i.e.

$$\int_B (\nabla u \cdot \nabla \varphi - g(u)\varphi) dx = 0$$

for all $\varphi \in C_0^{\infty}(B)$.

$$P(r) = r^{N-1}\left(rE(r) + \frac{N-2}{2}u(r)u'(r)\right),$$

(37)
$$\gamma_1 = \left(\frac{N}{p+1} - \frac{N-2}{2}\right) > 0, \quad \gamma_2 = \left(\frac{N}{q+1} - \frac{N-2}{2}\right),$$

$$\text{y} \quad \Gamma(u) = \begin{cases} \gamma_1 u^{p+1} & \text{para} \quad u \geq 0 \\ \gamma_2 |u|^{q+1} & \text{para} \quad u \leq 0. \end{cases}$$

Lemma 2.2. *If g is subcritical or sub-super subcritical, then there exists $b_1 \in \mathbb{R}$ such that for $b > b_1$ the solutions to (36) are singular and have no zero in $(0, \tilde{b})$.*

Proof. It is based on the fact that there is $m \in (0, 1)$ such that $P(m\tilde{b}) < 0$. Hence $P(t) < 0$ for all $t \in (0, m\tilde{b})$.

Lemma 2.3. *There exist $m_1 > 1$, $b_2 > b_1$, and $K > 0$ such that*

$$P(m_1\tilde{b}) \geq Ku^{p+1}(\tilde{b})\tilde{b}^N, \tag{38}$$

for $b > b_2$, uniformly for $a \in I$. In particular, $P(m_1\tilde{b}) \to +\infty$ as $b \to +\infty$.

Lemma 2.4. *There exists* $b_2 > b_1$ *such that if* $\hat{b} > b_1$, $\hat{a} \in I$, $u(\cdot, \hat{a}, \hat{b})$ *is a solution to* (36) *with* $u(1, \hat{a}, \hat{b}) = 0$, *then there exists* $\delta > 0$ *and continuous functions* $\alpha : (-\delta, \delta) \to \mathbb{R}$ *and* $\beta : (-\delta, \delta) \to \mathbb{R}$ *such that* $u(\cdot, \hat{a} + \alpha(t), \hat{b} + \beta(t))$ *satisfies* (18) *and* $u(\cdot, \hat{a} + \alpha(t), \hat{b} + \beta(t)) \neq u(\cdot, \hat{a} + \alpha(s), \hat{b} + \beta(s))$ *for* $s \neq t$. *Hence,* (18) *has a non-degenerate continuum of radial singular solutions.*

Let $\phi(r,a,b)$ be a differentiable function such that

$$\begin{aligned} u(r,a,b) &= \rho(r,a,b)\cos(\phi(r,a,b)),\\ u'(r,a,b) &= -\rho(r,a,b)\sin(\phi(r,a,b)),\\ \phi(\tilde{b},a,b) &= \tan^{-1}\left(\frac{-u'(\tilde{b},a,b)}{u(\tilde{b},a,b)}\right) \end{aligned} \tag{39}$$

By Lemma 2.3,

$$\lim_{b\to+\infty}\phi(1,a,b) = +\infty, \tag{40}$$

uniformly for $a \in I$. Hence, for each $a \in I$ there exists a positive integrer $J(a)$ and a sequence $\{b_j(a)\}_{j\geq J(a)}$ such that $\phi(1,a,b_j(a)) = j\pi + (\pi/2)$. By the continuous dependence on parameters of solutions to initial value problems, the functions b_j are continuous. By Lemma 2.4, $\{u(\cdot,a,b_j(a)); a \in I\}$ defines a non-degenerate continuum of singular solutions to (18).

These ideas extend to quasilinear equations. For example they can be taken to Kirchhoff's equation as follows.

Theorem 2.2 (Joint work with ShuZhi Song, [C-S, 2020]) *If* g *is supercubic and subcritical* $(3 < p, q < (N+2)/(N-2)\ )$ *then*

$$(41)\quad (a + b\int_B |\nabla u|^2)\Delta u(x) + g(u(x)) = 0,\ x \in B \quad u(x) = 0,\ x \in \partial B$$

has infinitely many radially symmetric solutions.

A LAPLACE-BELTRAMI EQUATION
shooting from singularity to singularity

Joint work with I. Ventura.
Let M be a codimension 1 manifold of revolution in $\mathbb{R}^N$ of class C^3 intersecting its axis of rotation. We assume M to be boundaryless, connected, and compact. Without loss of generality we may assume that M revolves around the $x_N = z$ axis. Also we may denote by $P_- = (0, \dots, -1)$ and $P_+ = (0, \dots, 1)$ the points of intersection of M with its axis of revolution. We study the existence of rotationally symmetric solutions to

$$\Delta_M u + f(u) = 0 \qquad \text{on } M, \tag{42}$$

where Δ_M is the *Laplace-Beltrami operator* on M and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that

$$\lim_{|u|\to+\infty} \frac{f(u)}{u} = +\infty. \tag{43}$$

Let $d : M \times M \to [0, \infty)$ denote the geodesic distance in M and $a = \max\{d(P_{-1}, x); x \in M\} = d(P_-, P_+)$. Hence there exist differentiable functions $G, z : [0, a] \to [0, \infty)$ such that $G(t) = 0$ if and only if $t \in \{0, a\}$, and $M = \{(\theta, z(r)); \theta = G(r), r \in [0, a]\}$. Moreover,

$$G'(0) = -G'(a) = 1, \quad z(0) = -1, \text{ and } z(a) = 1. \tag{44}$$

For $u : M \to \mathbb{R}$ rotationally symmetric the equation (42) is equivalent to

$$\begin{gathered} u_{tt} + (N-2)\frac{G'(t)}{G(t)}u_t + f(u(t)) = 0 \\ u(0),\ u(a) \in \mathbb{R}, \quad u'(0) = u'(a) = 0. \end{gathered} \tag{45}$$

We assume that there exists $m_1 > 0$ sych that

$$f \quad \text{increases on} \quad (-\infty, -m_1) \cup (m_1, +\infty). \tag{46}$$

Also we assume that there exist $p_1, p_2 \in (1, (N+1)/(N-3))$ such that

$$\begin{aligned} &\lim_{u\to+\infty} \frac{f(u)}{|u|^{p_1-1}u} := f_\infty \in (0,\infty), \quad \text{and} \\ &\lim_{u\to-\infty} \frac{f(u)}{|u|^{p_2-1}u} := f_{-\infty} \in (0,\infty). \end{aligned} \tag{47}$$

The main result is.
THEOREM 3.1: *The equation* (42) *has infinitely many rotationally symmetric solutions.*

In [C-F, 2015] for M a sphere it was proved that (42) has infinitely many rotationally symmetric solutions also symmetric with respect to the equator. That case becomes

$$
\begin{aligned}
u_{rr} + (N-2)\frac{\cos(r)}{\sin(r)} u_r + f(u) = 0, \quad r \in [0, \pi/2] \\
u'(0) = 0, \quad u'(\pi/2) = 0.
\end{aligned}
\tag{48}
$$

This equation is singular at 0 but not at $\pi/2$. That is the problem is very similar to the problem to finding radial solutions in a ball.

Let us consider the initial value problem

$$(49)\qquad \begin{aligned} &u_{tt} + (N-2)\frac{G'(t)}{G(t)}u_t + f(u(t)) = 0 \\ &u(0) = d,\ u'(0) = 0. \end{aligned}$$

Multiplying (49) by $G^{N-2}(t)$ we have

$$(50)\qquad (G^{N-2}(t)u_t)_t + G^{N-2}(t)f(u) = 0, \quad t \in [0, a]$$

For $\beta(t) = G^{N-2}(t) \int_{a/2}^{t} G^{2-N}(s)ds$,

$$\lim_{t \to 0} \beta(t) = 0. \tag{51}$$

Let $u(t,d) := u(t)$ be a solution to (49). Now the Pohozaev identity is

$$\begin{aligned} P(t,u) &:= G^{N-2}(t)\beta(t)\left[\frac{(u'(t))^2}{2} + F(u(t))\right] \\ &\quad - \frac{G^{N-2}(t)}{2} u(t)u'(t) \\ &= \int_0^t ((\beta(s)G^{N-2}(s))'F(u(s)) \\ &\qquad + \frac{G^{N-2}(s)}{2} u(s)f(u(s)))ds \end{aligned} \tag{52}$$

From (52) and (47) we deduce:

Lemma 3.1: *There exists $D > 0$ such that if $d > D$ then*

$$E(t,d) := \frac{(u'(t,d))^2}{2} + F(u(t,d)) > 0, \textit{ for all } \; t \in [0,a].$$

Moreover,

$$\lim_{|d| \to +\infty} E(t,d) = +\infty, \tag{53}$$

uniformly in t

Hence there exists a continuous function $\varphi : [0,a) \times [D,\infty) \to \mathbb{R}$ such that

$$u(t,d) = (u^2(t,d) + (u_t(t,d))^2)^{1/2} \cos(\varphi(t,d)),$$

$$u_t(t,d) = -(u^2(t,d) + (u_t(t,d))^2)^{1/2} \sin(\varphi(t,d)), \; \textit{and}$$

$$\lim_{d \to +\infty} \varphi(t,d) = +\infty, \; \textit{for each } t \in (0,a). \tag{54}$$

Lemma 3.2: *For each* $d \in \mathbb{R}$ *there exists* $M(d)$ *such that the solution to* (49) *satisfies* $|P(r,d)| \leq M(d)$ *for all* $r \in (0,a)$.
Lemma 3.3: *If* v *is bounded solution to* (49) *then* $\lim_{r \to a-} v'(r) = 0$.
Lemma 3.4: *For each* $d \geq D$, $u(\cdot, d)$ *has finitely many zeroes in* $[0,a)$.
Lemma 3.5: If $\lim_{t \to a} v(t, \hat{d}) = +\infty$, the there exists $\eta > 0$ such that if $|d - \hat{d}| < \eta$ then $\lim_{t \to a^-} v(t,d) = +\infty$.

Proof of Theorem 3.1. Let $d_0 > D$. Suppose that $u(\cdot, d_0)$ is not a solution to (45), i.e. does not define a solution to (42). By Lemmas 3.2 and 3.3, we may assume w.l.o.g. that $\lim_{t \to a-} u(t) = +\infty$ and that there exists $\epsilon > 0$ such that $u'(t, d_0) > 0$ for all $t \in (a - \epsilon, a)$. Let $\tilde{d} = \sup\{d \geq d_0; u(t, d)$ is monotonically increasing on $[t_1, a)$ and $\lim_{t \to a-} v(t, d) = +\infty\}$. Due to Lemma 3.5, $\tilde{d} > d_1$. By (54) $\tilde{d} < +\infty$. If $u(t, \tilde{d}) = 0$ for some $t \in (t_1, a)$, by continuous dependence there is a sequence $d_j \to \tilde{d}$, $d_j < \tilde{d}$ such that $u(t, d_j) \to 0$. Since this contradicts that $u(\cdot, d_j)$ increases on $[t_1, a)$, $u(\cdot, \tilde{d})$ is bounded. This and Lemma 3.3, prove that $u(\cdot, \tilde{d})$ is a solution to (45). Assuming that $u(\cdot, \tilde{d} + 1)$ is not a solution to (45) we find a second solution of the form $u(\cdot, \tilde{d}_1)$ with $\tilde{d}_1 \geq \tilde{d} + 1$. Iterating this process we have a sequence $\{\tilde{d}_k\}_k$ converging to $+\infty$ and such that $u(\cdot, \tilde{d}_k)$ is a solution to (45).

[A-C-C-2010] V. Ardila, J. Caicedo, and A. Castro, *Existence of non-degenerate continua of singular radial solutions for several classes of semilinear elliptic problems,* Milan J. Math. Vol. 77 (2010) 127150 DOI 10.1007/s00032-014-0226-6.

[B-C, 1988] A. Bahri and J.-M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain,* Comm. Pure Appl. Math. 41 (1988), no. 3, 253-294.

[C-C, 1994] A. Castro and J. Cossio, *Multiple radial solutions for a semilinear Dirichlet problem in a ball.* Proceedings of the Second Latin American Colloquium on Analysis (Bogot, 1992). Rev. Colombiana Mat. 27 (1993), no. 1-2, 15-24.

[C-C-H-V, 2020] A. Castro, J. Cossio, S. Herrón and C.A. Vélez, *Infinitely many radial solutions for a p-Laplacian problem with indefinite weight*, preprint.

[C-F, 2015] A. Castro and E.M. Fischer, *Infinitely Many Rotationally Symmetric Solutions to a Class of Semilinear Laplace-Beltrami Equations on Spheres*, Canadian Mathematical Bulletin, Vol 58 (2015), no. 4, 723-729.

[C-P-2010] A. Castro and V. Padrón, *Classification of radial solutions arising in the study of thermal structures with thermal equilibrium or no flux at the boundary*, Mem. Amer. Math. Soc. 208 (2010), no. 976.

[C-K-1987] A. Castro and A. Kurepa, *Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball,* Proc. Amer. Math. Soc. 101 (1987), no. 1, 57?64.

[C-K–1987] A. Castro and A. Kurepa, *Radially symmetric solutions to a superlinear Dirichlet problem with jumping nonlinearities,* Trans. Amer. Math. Soc. **315** (1989), pp. 353-372

[C-K-T-2007] A. Castro, J. Kwon, a C.M. Tan, *Infinitely many radial solutions for a sub-super critical Dirichlet boundary value problem in a ball*, Electronic Journal of Differential Equations, Vol. 2007(2007), No. 111 pp. 1-10.

[C-S-1989] A. Castro and R. Shivaji, *Non-negative solutions for a class of radially symmetric Non-positone problems*, Proc. Amer. Math. Soc. , Vol. 106, No. 3 (1989), pp. 735-740.

[C-S, 1989] A. Castro, and R. Shivaji, *Non-negative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric*, Comm. in Partial Differential Equations, 14, No. 8, (1989), pp. 1091-1100.

[C-S, 2020] A. Castro and ShuZhi Song, *Infinitely many radial solutions for a super-cubic Kirchhoff type problem in a ball*, Discrete and Continuous Dynamical Systems, doi:10.3934/dcdss.2020127.

[Ch-1967] S. Chandrasekhar, *Introduction to the study of stellar structure*, Dover Publications Inc., (1967).

[D-E-R-2002] J. Dolbeault, M. Esteban, and M. Ramaswamy, *Radial singular solutions of a critical problem in a ball,* Diff. Integral Equations 15 (2002), no. 12, 14591474.

[D-F-2007] J. Dolbeault and I. Florez, *Geometry of phase space and solutions semilinear equations in a ball,* Trans. Amer. Math. Soc., Vol. 359, No. 9, (2007), pp. 4073-4087.

[G-N-N, 1979] B. Gidas, W. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68, (1979), pp. 209-243.

[I-R-1995] M.H. Ibañez and P. Rosenzweig, *Analytical criteria for nonlinear instability of thermal structures*, Phys. Plamas 2 (11), (1995).

[J-L, 1973] D. Joseph and Lundgren, T., *Quasilinear Dirichlet problems driven by positive sources,* Arch. Rational Mech. Anal., 49, (1973), pp. 241-269.

[P-1989] D. Passaseo, *Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains,* Manuscripta Math. 65 (1989), no. 2, 147-165.

[Po-1965] S. I. Pohozaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR **165** (1965), pp. 36-39.