# On a problem of Graham on the p-divisibility of central binomial coefficients $\binom{2n}{n}$

Dr. Ernie Croot, Georgia Institute of Technology

February 4, 2022

**Abstract:** Graham offered \$1,000 to prove or disprove the existence of infinitely many integers $n$ such that $\binom{2n}{n}$ is coprime to $105 = 3 \cdot 5 \cdot 7$. In this talk I will present some heuristics concerning this conjecture, and will discuss recent work by myself, Hamed Mousavi, and Maxie Schmidt, showing the following: for every $r \geq 1$, and any $r$ distinct primes $p_1, p_2, ..., p_r \; \geq \; p_0(r)$, there exists an infinite sequence of integers $n_1, n_2, ...$ such that for all $j = 1, ..., r$, $\nu_{p_j}(\binom{2n_i}{n_i}) = o(\log n_i)$, where $\nu_p(n)$ denotes the number of times that a prime $p$ divides $n$. In other words, for any $r$ sufficiently large primes, there are infinitely many integers $n$ such that $\binom{2n}{n}$ is divisible by the primes $p_1, ..., p_r$ to only low multiplicity, since an upper bound for the number of times that a prime $p$ divides $\binom{2n}{n}$ is $\log n / \log p + 1$. This can be seen as a "statistical version" of Graham's Conjecture. It is more general in the sense that it says something about divisibility of $\binom{2n}{n}$ by not just 3 primes (like 3, 5, and 7), but any $r$ sufficiently large primes; but weaker in the sense that it doesn't show the existence of integers where $\binom{2n}{n}$ is coprime to $p_1 p_2 \cdot p_r$.