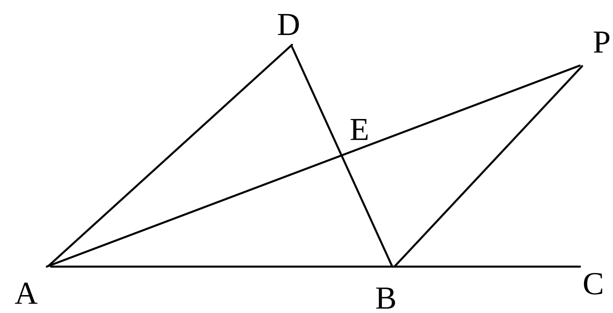
THE 2009-2010 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

1. An algebra teacher, testing her students on geometric sequences, asked her students to find the sum of all three terms of a geometric sequence in which the middle number was missing. Unfortunately, one student confused geometric sequences with arithmetic sequences, and then completed the problem with no other errors, obtaining an answer of 351. If all numbers involved were distinct positive integers, compute, with proof, the correct answer to the teacher's question.

2. The numbers $x_1, x_2, x_3, \ldots x_n$ are written on a chalkboard. A student chooses two of the numbers at random, *a* and *b*, erases them both, and writes just one number, *a* + *b* + *ab*, in their place. After 99 such operations only one number, N, is on the chalkboard. Prove that $N = (1 + x_1)(1 + x_2)(1 + x_3)\ldots(1 + x_n) - 1$.

3. If $\log_{\sin x}(\tan x) = \log_{\tan x}(\sin x)$, compute, with proof, all possible values of cos x.

4. In the diagram, $\overline{AP}$ bisects $\angle DAC$ and $\overline{BP}$ bisects $\angle DBC$. Find, with proof, the ratio of the measure of angle D to the measure of angle P.



5. A shirt has five large patches. Each patch covers at least $\frac{1}{2}$ of the shirt. Prove that there are 2 patches whose intersection covers at least $\frac{1}{5}$ of the shirt.

SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II 2009–10

1. Represent the terms of the geometric series with a, ar, ar^2. Then the student's sequence becomes a, $\frac{1}{2}(a+ar^2)$, ar^2. Thus, $a+\frac{1}{2}(a+ar^2)+ar^2=351$.

   Simplifying, $3a+3ar^2=702$. Solving this last equation for a, $a=\frac{234}{1+r^2}$.
   Noting that 234 = (2)(3^2)(13), this equation will only yield integral values of a for r = 1 and r = 5. If r = 1, the numbers in the teacher's sequence are not distinct. Therefore, r = 5 and the three numbers are 9, 45, and 225 with a sum of 279.

2. Proof by mathematical induction on the number of numbers, n.
   If n =1, we certainly have $(1+x_1)-1=x_1$

   Suppose the statement is true for all k < n numbers. Suppose the process has been repeated until only two numbers, *a* and *b*, are left, and suppose *a* was obtained by combining $x_1, x_2, \ldots x_m$ and *b* was obtained by combining $x_{m+1}, x_{m+2}, \ldots x_k$. Then by the induction hypothesis,

   $$a=(1+x_1)(1+x_2)\ldots(1+x_m)-1 \text{ and } b=(1+x_{m+1})(1+x_{m+2})\ldots(1+x_k)-1.$$

   Therefore, the final number is given by a + b + ab =

$$a+b+(1+x_1)(1+x_2)\ldots(1+x_k)-\underbrace{(1+x_1)(1+x_2)\ldots(1+x_m)}-\underbrace{(1+x_{m+1})(1+x_{m+2})\ldots(1+x_k)}+1=$$

$$a+b+(1+x_1)(1+x_2)\ldots(1+x_k)-(a+1)-(b+1)+1=$$

$$(1+x_1)(1+x_2)(1+x_3)\ldots(1+x_k)-1$$

3. $\log_{\sin x}(\tan x)=\log_{\tan x}(\sin x)=\frac{1}{\log_{\sin x}(\tan x)}$ (by the change of base formula),

   where sin x > 0 and tan x > 0. Therefore, $(\log_{\sin x}(\tan x))^2=1$,

   which implies $\log_{\sin x}(\tan x)=\pm 1$.

   $\log_{\sin x}(\tan x)=1 \longrightarrow \tan x=\sin x \longrightarrow \cos x=1.$

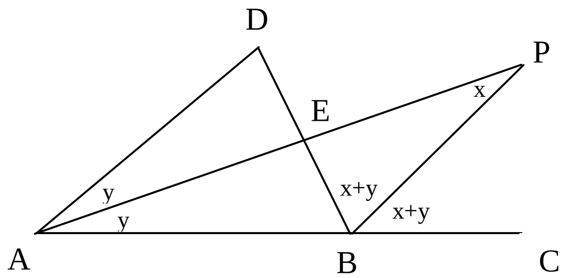
   However, this would make sin x = 0.

   $\log_{\sin x}(\tan x)=-1 \longrightarrow \tan x=\frac{1}{\sin x} \longrightarrow \cos x=\sin^2 x=1-\cos^2 x$

   Therefore, $\cos^2 x+\cos x-1=0 \longrightarrow \cos x=\frac{-1\pm\sqrt{5}}{2}$.

   If cos x is negative, then one of tan x or sin x is negative also. Therefore, the only possible value of cos x is $\frac{-1+\sqrt{5}}{2}$.

4. Let m∠P = x and m∠PAB = m∠PAD = y.
Since ∠PBC is an exterior angle of triangle PAB,
m∠PBC= x+y and m∠DBP= x+y also.
Therefore, m∠DBC= 2x+2y. Since ∠DBC is an exterior angle of triangle ABD,
m∠DBC = m∠D + m∠DAB or
2x + 2y = m∠D + 2y.
Therefore, m∠D = 2x and the desired ratio is 2:1.



5. Let the area of the shirt T be equal to 1.
Denote by Aj the j-th patch and by Pj = |Aj| the area of j-th patch.
Further we denote by Pij=|Ai ∩ Aj| the area of intersection of the i-th patch and j-th patch.
Analogously we define Pijk, Pijkl, and P_{12345}

Note that Pj ≥1/2.

Using the inclusion-exclusion principle (or sieve principle)

$$1 = T \geq \left|\bigcup_{j=1}^{5} A_j\right| = \sum P_j - \sum P_{ij} + \sum P_{ijk} - \sum P_{ijkl} + P_{12345} \Rightarrow$$
$$1 - \sum P_j + \sum P_{ij} - \sum P_{ijk} + \sum P_{ijkl} - P_{12345} \geq 0 \; (*)$$

For every patch estimate Pj using the inclusion-exclusion principle:

$$P_1 \geq |A_{12} \cup A_{13} \cup A_{14} \cup A_{15}| = \sum P_{1j} - \sum P_{1jk} + \sum P_{1jkl} - P_{12345}$$
$$P_2 \geq |A_{21} \cup A_{23} \cup A_{24} \cup A_{25}| = \sum P_{2j} - \sum P_{2jk} + \sum P_{2jkl} - P_{12345}$$
$$P_3 \geq |A_{32} \cup A_{31} \cup A_{34} \cup A_{35}| = \sum P_{3j} - \sum P_{3jk} + \sum P_{3jkl} - P_{12345}$$
$$P_4 \geq |A_{42} \cup A_{43} \cup A_{41} \cup A_{45}| = \sum P_{4j} - \sum P_{4jk} + \sum P_{4jkl} - P_{12345}$$
$$P_5 \geq |A_{52} \cup A_{53} \cup A_{54} \cup A_{51}| = \sum P_{5j} - \sum P_{5jk} + \sum P_{5jkl} - P_{12345}$$

Adding above inequalities we obtain:

$$\sum P_j \geq 2\sum P_{ij} - 3\sum P_{ijk} + 4\sum P_{ijkl} - 5P_{12345} \Rightarrow \sum P_j - 2\sum P_{ij} + 3\sum P_{ijk} - 4\sum P_{ijkl} + 5P_{12345} \geq 0 \; (**)$$

Multiplying (**) by 1/3 and adding it to (*) we get

$$1 - \frac{2}{3}\sum P_j + \frac{1}{3}\sum P_{ij} - \frac{1}{3}\sum P_{ijkl} + \frac{2}{3}P_{12345} \geq 0 \; (***)$$

Since $P_{ijkl} \geq P_{12345}$ and there are 5 of those, we have

$$\sum P_{ijkl} \geq 5P_{12345} \Rightarrow \frac{1}{3}\sum P_{ijkl} - \frac{2}{3}P_{12345} \geq 0$$

Hence

$$1 - \frac{2}{3}\sum P_j + \frac{1}{3}\sum P_{ij} \geq \frac{1}{3}\sum P_{ijkl} - \frac{2}{3}P_{12345} \geq 0$$

This implies

$$\sum P_{ij} \geq 2\sum P_j - 3 \geq 2\left(5\frac{1}{2}\right) - 3 = 2$$

Since there are $\binom{5}{2} = 10$ summands in $\sum P_{ij}$ there must be one (pigeonhole principle) which is larger than 2/10.