1. A and B both represent nonzero digits (not necessarily distinct). If the base ten numeral \( A B \) divides, without remainder, the base ten numeral \( A 0 B \) (whose middle digit is zero), find, with proof, all possible values of \( A B \).

2. A and B are points on the positive x and positive y axes respectively and C is the point with coordinates (3, 4). Prove that the perimeter of triangle ABC is greater than 10.

3. One solution for the equation \( a^2 + b^2 + c^2 + 2 = abc \) is \( a = 3, b = 3 \) and \( c = 4 \).
   a. Find a solution \((a, b, c)\) where \(a, b,\) and \(c\) are integers all larger than 10.
   b. Prove that there are infinitely many solutions \((a, b, c)\) where \(a, b,\) and \(c\) are positive integers.

4. Consider the equation \( \sqrt{x} = \sqrt{a} + \sqrt{b} \), where \(x\) is a positive integer.
   a. Prove that the equation has a solution \((a, b)\) where \(a\) and \(b\) are both positive integers, if and only if \(x\) has a factor which is a perfect square greater than 1.
   b. If \(x \leq 1,000\), compute, with proof, the number of values of \(x\) for which the equation has at least one solution \((a, b)\) where \(a\) and \(b\) are both positive integers.

5. In right triangle ABC, AC = 6, BC = 8 and AB = 10. PA and PB bisect angles A and B respectively. Compute, with proof, the ratio \( \frac{PA}{PB} \).
1. Of course, this problem can be done by trial and error (there are only 81 possibilities), but we present a more elegant solution.

Suppose \( \frac{A0B}{AB} = k \). Then \( 100A + B = 10Ak + Bk \) or

\[
(i) \quad 100A - 10Ak = Bk - B = B(k - 1)
\]

Since the left side of equation (i) is a multiple of 5, the right side must also be. Since \( 1 < k < 10 \), the right side is positive and thus so is the left side. Then either 5 divides \( k - 1 \) or 5 divides \( B \).

Suppose 5 divides \( k - 1 \). Then \( k = 6 \), so that (i) becomes \( 40A = 5B \), or \( B = 8A \). Therefore, \( A = 1 \), \( B = 8 \), and \( AB = 18 \).

Now suppose 5 divides \( B \). Then \( B = 5 \), and (i) becomes \( 10A(10 - k) = 5(k - 1) \), or \( 2A(10 - k) = k - 1 \). From this, \( A = \frac{k - 1}{2(10 - k)} \). Since the denominator is even, \( k - 1 \) must be even and \( k \) is odd. Trying \( k = 3, 5, 7, \) and 9, we find only \( AB = 15 \) and 45 corresponding to \( k = 7, 9 \) respectively. Therefore, the only possible values for \( AB \) are 15, 18, and 45.

2. Consider the reflection images of \( C \) over the \( x \) and \( y \) axes. Call these points \( C_x \) and \( C_y \), respectively, as shown. The coordinates of \( C_x \) are (3, -4) and of \( C_y \) are (-3, 4). The length of \( CC_x \) is \( 2(4) = 8 \) and the length of \( CC_y \) is \( 2(3) = 6 \).

Since \( \Delta ABC \) is a right triangle, the length of \( CC_x \) is \( \sqrt{6^2 + 8^2} = 10 \). Also note that because \( C_y \) is a reflection image of \( C \), \( BC = BC_y \). Similarly, \( AC = AC_x \).

In quadrilateral \( ABC_yC_x \), \( C_yB + BA + AC_x > CC_y = 10 \), Therefore, by substitution, \( BC + BA + AC > 10 \).
3. Suppose we begin with two positive integers $a$ and $b$, and we try to find a third integer $x$ such that $a^2 + b^2 + x^2 + 2 = abx$. Then the problem can be thought of as finding an integer solution (if one exists) for the quadratic equation $x^2 - (ab)x + (a^2 + b^2 + 2) = 0$.

If there is some integer solution $x = c$, then there must exist a real number $d$ such that

$$x^2 - (ab)x + (a^2 + b^2 + 2) = (x-c)(x-d) = x^2 - (c+d)x + cd$$

Comparing the coefficients on the left and right sides of this last equation, we know that $ab = c + d$, so that $d = ab - c$ is also an integer. Therefore, given any three integers $a$, $b$, and $c$ such that $a^2 + b^2 + c^2 + 2 = abc$, we can replace $c$ with $ab - c$ to obtain another solution.

We know that $(4, 3, 3)$ is a solution. So we can replace one of the 3’s with $3 \cdot 4 - 3 = 9$ to get the solution $(4, 3, 9)$. Since $a$, $b$, and $c$ are interchangeable, we can obtain other solutions by repeatedly replacing the smallest number (which we will call $c$) by $ab - c$. Hence, listing the numbers in decreasing order at each step, we obtain the following solutions:

$$(4, 3, 3) \rightarrow (9, 4, 3) \rightarrow (33, 9, 4) \rightarrow (293, 33, 9) \rightarrow (9660, 293, 33).$$

Since this process can be repeated indefinitely, there are infinitely many positive integer solutions $(a, b, c)$ to the given equation.
4. (i) Given \( \sqrt{x} = \sqrt{a} + \sqrt{b} \).

Suppose \( x = k^2 y \), with \( k \) and \( y \) positive integers, and \( k > 1 \). We must prove that there exists at least one pair of positive integers \((a, b)\) that satisfies the equation.

We have \( \sqrt{x} = \sqrt{k^2 y} = k \sqrt{y} \). Since \( k > 1 \), then \( k - 1 > 0 \). Therefore,

\[
\sqrt{x} = k \sqrt{y} = (k - 1) \sqrt{y} + \sqrt{y} = \sqrt{(k - 1)^2 y + y}.
\]

Since both \((k - 1)^2 y \) and \( y \) are both positive integers, setting \( a = (k - 1)^2 y \) and \( b = y \) gives the desired result.

Now suppose \( a \) and \( b \) are both positive integers that satisfy \( \sqrt{x} = \sqrt{a} + \sqrt{b} \).

We must show that \( x \) has a perfect square factor greater than 1.

\[
\sqrt{x} = \sqrt{a} + \sqrt{b} \quad \Rightarrow \quad x = (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}
\]

Since \( x \) is a positive integer, \( \sqrt{ab} \) must be a perfect square. There are two possibilities: either (1) \( a \) and \( b \) are both perfect squares or (2) the non-square factors of \( a \) and \( b \) are equal.

1) If \( a \) and \( b \) both perfect squares, let \( a = m^2 \) and \( b = n^2 \). Then

\[
x = a + b + 2\sqrt{ab} = m^2 + n^2 + 2mn = (m + n)^2.
\]

Therefore, \( x \) has a perfect square factor.

2) If the non-square factors of \( a \) and \( b \) are equal, let \( a = m^2 p \) and \( b = n^2 p \). Then

\[
x = a + b + 2\sqrt{ab} = m^2 p + n^2 p + 2mnp = p(m + n)^2
\]

and again, \( x \) has a perfect square factor.

Therefore, the equation has a solution \((a, b)\) where \( a \) and \( b \) are both positive integers, if and only if \( x \) has a factor which is a perfect square greater than 1.

(ii) There are 250 values of \( x \leq 1000 \) that contain a factor of 4. Similarly, the number of values of \( x \leq 1000 \) that, respectively, contain a factor of \( 3^2, 5^2, 7^2, 9^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2 \) is 111, 40, 20, 8, 5, 3, 2, 1, 1, and 1, for a total of 442. However, some values, like \( 36 = (2^2)(3^2) \), have been counted twice and must be subtracted from our total. The number of values of \( x \leq 1000 \) that, respectively, contain a factor of \( (2^2)(3^2), (2^2)(5^2), (2^2)(7^2), (2^2)(11^2), (2^2)(13^2), (3^2)(5^2), \) and \( (3^2)(7^2) \)

is 27, 10, 5, 2, 1, 4, and 2, a total of 51 such duplicates. However, the factor \( (2^2)(3^2)(5^2) \)

was counted three times, once in each group. Therefore, the final total is

\[
442 - 51 + 1 = 392.
\]
5. Method 1:

We will refer to \( \angle CAB \) as \( \angle A \) and \( \angle CBA \) as \( \angle B \). So that \( m \angle A + m \angle B = 90^\circ \).

Then \( m \angle P = 180 - \frac{1}{2} (m \angle A + m \angle B) = 135^\circ \).
So that, \( m \angle PAB + m \angle PBA = 45 \). Represent the measures of these two angles with \( \alpha \) and \( 45 - \alpha \).

Using the Law of Sines on \( \triangle APB \)

\[
\frac{PA}{\sin(45 - \alpha)} = \frac{PB}{\sin \alpha} = \frac{\sin 45 \cos \alpha - \cos 45 \sin \alpha}{\sin \alpha} = \sin 45 \cot \alpha - \cos 45.
\]

Now \( \cot \alpha = \cot (\frac{1}{2}A) = \frac{1 + \cos A}{\sin A} \) (using the appropriate half-angle formula)

But in \( \triangle ABC \), \( \cos A = \frac{6}{10} \) and \( \sin A = \frac{8}{10} \), making \( \cot \alpha = \frac{1 + 6}{8} = 2 \).

Finally, \( \frac{PA}{PB} = (\sin 45)(2) - \cos 45 = \frac{\sqrt{2}}{2}(2) - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \).

Method 2:

Note that since point P is the intersection of the angle bisectors of \( \triangle ABC \), P is the incenter (the center of the inscribed circle).

Noting that the tangent segments to a circle from an external point are congruent, represent the lengths of the segments in the diagram as shown.

Then \( 6 - x + 8 - x = 10 \) and \( x = 2 \).
Therefore, right \( \triangle ARP \) has side lengths 2, 4, and \( 2\sqrt{5} \), and right \( \triangle BMP \) has side lengths \( 2, 6, \) and \( 2\sqrt{10} \).

Therefore, \( \frac{PA}{PB} = \frac{2\sqrt{5}}{2\sqrt{10}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \).