1. The equation \( y = x^2 + 2ax + a \) represents a parabola for all real values of \( a \).

(a) Prove that all of these parabolas pass through a common point and determine the coordinates of this point.

(b) The vertices of all the parabolas lie on a curve. Find, with proof, the equation of this curve.

2. A sequence of functions is defined by the following rules

\[
\begin{align*}
(i) & \quad f_1(x) = \frac{2x-1}{x+1} \\
(ii) & \quad f_{n+1}(x) = f_1(f_n(x))
\end{align*}
\]

for \( n = 1, 2, 3, \ldots \). Compute, with proof, \( f_{2012}(2013) \).

3. Consider the three numbers \( 6n^2 + 5 \), \( 2n^2 + 3 \), and \( n^2 + 1 \), where \( n \) is a positive integer.

(a) Find, with proof, all values of \( n \) for which all three numbers are prime numbers.

(b) Prove that there are infinitely many values of \( n \) for which none of the three numbers is a prime number.

4. Six hotel guests wanted to check out, but the front desk clerk was nowhere to be found. The guests each put their room key on the counter and left. When the clerk returned, he did not know which of the 6 keys went to which room. If the clerk randomly gave the keys to each of the next six guests, compute, with proof, the probability that none of the new guests received the correct room key.

5. In the figure, ABCD is a square, M is the midpoint of AB, and N is the midpoint of BC. AN and CM intersect at point P. Compute, with proof, the ratio of the area of quadrilateral APCD to the area of square ABCD.

Calculated are NOT permitted.
1. a) We want to show that there is a point \((x_1, y_1)\) which passes through all parabolas with equations of the form \(y = x^2 + 2ax + a\). Let \(m\) and \(n\) be two distinct real numbers. Then we want
\[
y_1 = x_1^2 + 2mx_1 + m \quad \text{and} \quad y_1 = x_1^2 + 2nx_1 + n.
\]
Subtracting the second equation from the first gives
\[
2mx_1 - 2nx_1 + (m - n) = 0 \quad \Rightarrow \quad 2x_1(m - n) = -(m - n).
\]
Since \(m \neq n\), we must have \(x_1 = -\frac{1}{2}\), so \(y_1 = \frac{1}{4}\). Thus the desired point is \((-\frac{1}{2}, \frac{1}{4})\).

b) To find the vertex of each parabola \(y = x^2 + 2ax + a\), we complete the square, giving \(y = x^2 + 2ax + a^2 + a - a^2 = (x + a)^2 + (a - a^2)\). Therefore, the vertex has coordinates \((-a, a - a^2)\). Thus, the vertices lie on a parabola with equation \(y = f(x) = -x^2 - x\).

2. Let’s see if there is a pattern to the sequence.
\[
f_1(x) = \frac{2x-1}{x+1}, \quad f_2(x) = f_1(f_1(x)) = \frac{2\left(\frac{2x-1}{x+1}\right) - 1}{2x-1 + 1} = \frac{x-1}{x}. \quad \text{In a similar manner we find,}
\]
\[
f_3(x) = f_1(f_2(x)) = \frac{2\left(\frac{x-1}{x}\right) - 1}{x-1 + 1} = \frac{x-2}{2x-1}, \quad \text{Similarly,} \quad f_4(x) = \frac{1}{1-x}, \quad f_5(x) = \frac{x+1}{2-x},
\]
\[
f_6(x) = x, \quad \text{and} \quad f_7(x) = \frac{2x-1}{x+1}. \quad \text{So we see that} \quad f_{6n}(x) = x \quad \text{for} \quad n = 1, 2, 3, \ldots.
\]
Since 2012 = (6)(335) + 2, we have \(f_{2012}(2013) = f_2(2013) = \frac{2013-1}{2013} = \frac{2012}{2013}\).
3. By inspection, if \( n = 1 \) the three numbers are 11, 5, and 2, all prime. If \( n = 2 \), the three numbers are 29, 11, and 5, again all prime. However when \( n = 3 \), we get \{59, 21, 10\}, when \( n = 4 \) we get \{101, 35, 17\}, and when \( n = 5 \) we get \{155, 53, 26\}, so that each set consists of at least one non-prime. More specifically, each of the sets of three numbers listed above contains a multiple of 5.

(a) We will prove that one of the three numbers must be a multiple of 5 for \( n > 2 \).

Any integer must have one of the forms \( 5k \), \( 5k \pm 1 \), or \( 5k \pm 2 \).

If \( n = 5k \), \( 6n^2 + 5 = 150k^2 + 5 = 5(30k^2 + 1) \) which is clearly a multiple of 5.

If \( n = 5k \pm 1 \), then \( 2n^2 + 3 = 2(5k \pm 1)^2 + 3 = 50k^2 \pm 20k + 5 = 5(10k^2 \pm 4k + 1) \), again a multiple of 5.

If \( n = 5k \pm 2 \), then \( n^2 + 1 = (5k \pm 2)^2 + 1 = 25k^2 \pm 20k + 5 = 5(4k^2 \pm 4k + 1) \), again a multiple of 5.

In all three cases, therefore, one of the three numbers \( 6n^2 + 5 \), \( 2n^2 + 3 \), and \( n^2 + 1 \) is a multiple of 5. Since each number is larger than 5 for \( n > 2 \), the only values of \( n \) for which all three numbers are prime are \( n = 1 \) and \( n = 2 \).

(b) If \( n \) is an odd multiple of 5, \( 6n^2 + 5 \) is a multiple of 5, and \( n^2 + 1 \) is even.

If \( n \) is an odd multiple of 3, \( 2n^2 + 3 \) is a multiple of 3, and \( n^2 + 1 \) is even.

Hence, if \( n \) is any odd multiple of 15, \( 6n^2 + 5 \) is a multiple of 5 (and therefore not prime), \( 2n^2 + 3 \) is a multiple of 3 (and therefore not prime), and \( n^2 + 1 \) is even (and therefore not prime). Therefore, there are infinitely many values of \( n \) for which none of the three numbers is prime.
4. It is easier to compute the probability of the opposite event, namely the one where at least one of the new guests receives the correct room key. Denote that probability by \( p(B) \). If the desired probability is \( p(A) \), then we compute it as \( p(A) = 1 - p(B) \).

Let \( A_k \) = probability that guest \( k \) received the correct room key, \( A_{kl} \) = probability that guests \( k \) and \( l \) got the correct room keys and so on analogously. For example \( A_{235} \) denotes the probability that guests 2,3 and 5 received the correct room key.

Using the principle of inclusion-exclusion we have

\[
p(B) = \sum_{k=1}^{6} A_k - \sum_{k=1, l>k}^{6} A_{kl} + \sum_{k=1, l>k, m>l}^{6} A_{klm} - \sum_{k=1, l>k, m>l, n>m} A_{klmn} + \sum_{k=1, l>k, m>l, n>m} A_{klmn} - \sum_{k=1, l>k, m>l, n>m} A_{klmn}.
\]

Notice that \( A_1 = A_2 = A_3 = A_4 = A_5 = A_6 \) and that \( A_{kl} = A_{mn} \) for all choices of \( k, l, m, n \), and so on. Hence,

\[
p(B) = 6 A_1 - \binom{6}{2} A_{12} - \binom{6}{3} A_{123} - \binom{6}{4} A_{1234} - \binom{6}{5} A_{12345} - \binom{6}{6} A_{123456}.
\]

Now we compute

\[
A_1 = \frac{6!}{6!}, A_{12} = \frac{4!}{6!}, A_{123} = \frac{3!}{6!}, A_{1234} = \frac{2!}{6!}, A_{12345} = \frac{1!}{6!}, A_{123456} = \frac{0!}{6!}.
\]

Therefore,

\[
p(B) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}.
\]

It follows

\[
p(A) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \frac{265}{720} = \frac{53}{144} \approx 0.368.
\]

5. Method 1

Construct diagonal \( \overline{AC} \). Since \( \overline{CM} \) and \( \overline{AN} \) are medians of triangle \( \triangle ABC \), point \( P \) is the centroid of triangle \( \triangle ABC \), so that \( PM = \frac{1}{3} CM \).

Construct a perpendicular from \( P \) to \( \overline{AB} \). \( \triangle MBC \) and \( \triangle MQP \) are similar, \( PQ = \frac{1}{2} CB \).

Since \( \overline{PQ} \) and \( \overline{CB} \) are the altitudes of \( \triangle ABP \) and \( \triangle ABC \), respectively, and they have the same base \( \overline{AB} \), the area of \( \triangle ABP \) is \( \frac{1}{6} \) the area of \( \triangle ABC \).

Thus, area of \( \triangle ABP = \frac{1}{6} \left( \frac{1}{2} AB^2 \right) = \frac{1}{6} AB^2 \).

Similarly, the area of \( \triangle CBP = \frac{1}{6} (AB^2) \).

Therefore, Area of quad \( APCD = \) Area of \( ABCD - (\text{area of } ABP - \text{area of } \triangle CBP) = AB^2 - [\frac{1}{6} (AB^2) + \frac{1}{6} (AB^2)] = \frac{1}{6} (AB^2) \). Therefore, the desired ratio is \( \frac{2}{3} \).
Method 2

Construct diagonal $\overline{AC}$. Let $m\angle CMB = m\angle ANB = \alpha$ and $m\angle MPN = m\angle APC = \theta$. Using quadrilateral BMPN, $\theta = 270 - 2\alpha$.

$\sin\theta = \sin(270 - 2\alpha) = \sin270\cos2\alpha - \cos270\sin2\alpha = -\cos2\alpha$

Without loss of generality, let $CB = 2$, so that $MB = 1$ and $CM = \sqrt{5}$.

Using right $\Delta MBC$, $\sin\alpha = \frac{2}{\sqrt{5}}$. Therefore, using the appropriate double angle formula, $\cos2\alpha = 1 - 2\sin^2\alpha = 1 - 2\left(\frac{2}{\sqrt{5}}\right)^2 = -\frac{3}{5}$.

Therefore $\sin\theta = \frac{3}{5}$

Since $\overline{CM}$ and $\overline{AN}$ are medians of triangle $ABC$, point $P$ is the centroid of the triangle, so that $AP = CP = \frac{1}{3} CM = \frac{1}{3} \sqrt{5}$.

Therefore, the area of $\Delta APC = \frac{1}{2} (AP)(PC)\sin\theta = \frac{1}{2} \left(\frac{1}{3} \sqrt{5}\right)\left(\frac{2}{\sqrt{5}}\right)\left(\frac{3}{5}\right) = \frac{2}{3}$.

The area of $\Delta ADC = \frac{1}{2} (2)^2 = 2$, and the area of quad $APCD = 2 + \frac{2}{3} = \frac{8}{3}$.

Therefore, the desired ratio is $\frac{3}{4} = \frac{2}{3}$.

Method 3

Place the figure in the coordinate plane so the vertices of the square have the following coordinates: $A = (0, 0), B = (1, 0), C = (1, 1), D = (0, 1)$.

Then, segment $AN$ falls on the line $y = \frac{1}{2}x$ and segment $CM$ falls on the line $y = 2x - 1$

Since $\frac{1}{2}x = 2x - 1 \Rightarrow x = \frac{2}{3}$, the coordinates of $P$ are $(2/3, 1/3)$

Draw a horizontal line through $P$ that intersects $AD$ at $E = (0, 1/3)$ and $BC$ at $F = (1, 1/3)$

Since triangle $AEP$ is congruent to triangle $CFP$, the area of quadrilateral $APCD$ equals the area of rectangle $CDEF$.

$CDEF$ has an area of $2/3$ and the area of $ABCD$ is $1$. So, $2/3$ is the desired ratio.